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On total ϕ_0-stability of nonlinear systems of differential equations

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Abstract

The notions of ϕ_0-stability of systems of ordinary differential equations (ODEs) were introduced. In this paper, we will extend the ϕ_0-stability notion to a new type of stability called total ϕ_0-stability, and give some criteria and results. Our technique depends on Liapunov's direct method.

Keywords: ϕ_0-Equistability; Uniform ϕ_0-stability; Uniform asymptotic ϕ_0-stability; Uniform total ϕ_0-stability

1. Introduction

The problems of the qualitative properties of differential equations has been successfully studied in different approaches based on Liapunov's direct method, such as cone and cone-valued Liapunov function method (see [3]).

Consider the system

$$x' = f(t,x), \tag{1.1}$$

and the perturbed system

$$x' = f(t,x) + h(t,x), \tag{1.2}$$

where $f, h \in C[J \times \mathscr{R}^n, \mathscr{R}]$, $J = [t_0, \infty)$ and $f(t,0) = h(t,0) = 0$, with $x(t_0, t_0, x_0) = x_0$, $\mathscr{R}^n$ is the n-dimensional Euclidean real space, $\mathscr{R} = (-\infty, \infty)$. Define

$$S_\rho = \{x, x \in \mathscr{R}^n,\ \|x\| < \rho,\ \rho > 0\}.$$

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The aim of this paper is to extend the notions of ϕ_0-stability of [1] to the so-called total ϕ_0-stability of the systems (1.1) and (1.2). These notions in the case of uniformly lie somewhere between uniform ϕ_0-stability of [1] on one side, and uniform total stability of [2] on the other side.

Going through [1], we shall investigate these notions and obtain the necessary conditions to construct cone-valued Liapunov function.

Now as in [3], we define a Liapunov function $V(t,x) \in C[J \times \mathscr{R}^n, \mathscr{R}]$ and the function

$$D^+V(t,x) = \lim_{\delta\to 0^+} \sup \frac{1}{\delta}[V(t+\delta, x+\delta f(t,x)) - V(t,x)].$$

The following definitions will be needed.

Definition 1.1 [2]. A function $\phi(r)$ is said to belong to the class $\mathscr{K}$ if $\phi(r) \in C[(0,\rho), R^+], \phi(0) = 0$ and $\phi(r)$ is strictly monotone increasing in r.

Definition 1.2 [2]. A function $\psi(t)$ is said to belong to the class $\mathscr{L}$ if $\psi(t) \in C[J, R^+], \psi(t) \to \infty$ and $\psi(t)$ is strictly monotone decreasing in t.

Definition 1.3 [1]. A proper subset K of $\mathscr{R}^n$ is called a cone if:

(i) $\lambda K \subset K,\ \lambda \geqslant 0$;
(ii) $K + K \subset K$;
(iii) $\overline{K} = K$;
(iv) $K^\circ \neq \emptyset$ and
(v) $K \cap (-K) = \{0\}$,

where $\overline{K}$ and K° denote the closure and interior of K, respectively, and ∂K denotes the boundary of K.

The order relation on $\mathscr{R}^n$ induced by the cone K is defined as follows. Let $x, y \in K$. Then

$$x \leqslant_K y \quad \text{iff} \quad y - x \in K \quad \text{and} \quad x <_{K^\circ} y \quad \text{iff} \quad y - x \in K^\circ.$$

The set

$$K^* = \{\phi \in \mathscr{R}^n\colon (\phi, x) \geqslant 0,\ x \in K\}$$

is called the adjoint cone if it satisfies properties (i)–(v) of Definition 1.3, $x \in \partial K$ iff $(y, x) = 0$ for some $y \in K_0^*$, $K_0 = K \setminus \{0\}$.

Definition 1.4 [1]. A function $L : D \to \mathscr{R}^n,\ D \subset \mathscr{R}^n$, is called *quasimonotone* relative to the cone K if $x, y \in D$ and $y - x \in \partial K$, then there exists $\phi_0 \in K_0^*$ such that $(\phi_0, y - x)$ and $(\phi_0, L(y) - L(x)) \geqslant 0$.

Following [1], we define the set

$$S(\rho) = \{x \in K : \|x\| \leqslant \rho,\ \ \rho > 0\}.$$

Definition 1.5 [1]. The zero solution of (1.1) is said to be ϕ_0-*equistable* if for each $\epsilon > 0$, there exists $\delta = \delta(t_0, \epsilon)$ continuous in t_0, for each ϵ, such that the inequality

$$(\phi_0, x_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t)) < \epsilon,\ \ t \geqslant t_0,$$

where here and in the rest of this paper $x^*(t)$ denotes the maximal solution of (1.1) relative to the cone $K \subset \mathscr{R}^n$.

Other ϕ_0-stability concepts can be similarly defined (see [1]).

Definition 1.6. The zero solution of (1.1) is said to be totally ϕ_0-stable if, for every $\epsilon > 0, t_0 \in \mathscr{R}$, there exist two constants $\delta_1 = \delta(t_0, \epsilon)$, $\delta_2(t_0, \epsilon)$ such that for the maximal solution $x^*(t)$ of (1.2) and $\phi_0 \in K_0^*$, the inequality

$$(\phi_0, x^*(t)) < \epsilon \quad \text{for } t \geqslant t_0$$

provided that

$$(\phi_0, x_0) < \delta_1 \quad \text{and} \quad (\phi_0, h(t,x)) < \delta_2.$$

Definition 1.7. The zero solution of (1.1) is said to be totally ϕ_0-stable under permanent perturbations bounded in the mean if for every $\epsilon > 0, t_0 \in \mathscr{R}^+$, and $T > 0$ there exist two positive constants $\delta_1 = \delta_1(\epsilon)$ and $\delta_2 = \delta_2(t_0, \epsilon)$ such that for every solution $x(t, t_0, x_0)$ of the perturbed system (1.2), the inequality

$$(\phi_0, x^*(t)) < \epsilon \quad \text{for } t \geqslant t_0$$

provided that

$$(\phi_0, x_0) < \delta_1 \quad \text{and} \quad (\phi_0, h(t,x)) < \gamma(t)$$

and

$$\int_{t_0}^{t_0+T} \gamma(s)\mathrm{d}s < \delta_2.$$

Definition 1.8. The zero solution of (1.1) is said to be uniformly totally ϕ_0-stable if a solution $x(t, t_0, x_0)$ of (1.2) is uniformly asymptotically ϕ_0-stable with $h(t, 0) = 0$, and $(\phi_0, h(t,x)) \leqslant \sigma(t), \quad \sigma \in \mathscr{L}$.

2. Main results

In this section, we will discuss and obtain some results of the total ϕ_0-stability of the system (1.1).

Theorem 2.1. *Let the zero solution of* (1.1) *be uniformly asymptotically ϕ_0-stable. Assume further that*

$$\|f(t,x) - f(t,y)\| \leqslant L(t)\|x - y\|,$$

for $(t,x), (t,y) \in \mathscr{R}^+ \times K$, $L(t) \geqslant 0$ is a continuous function defined on $\mathscr{R}^+$, and

$$\left| \int_{t_0}^{t_0+T} L(s)\mathrm{d}s \right| \leqslant \alpha T, \alpha \text{ is a constant.}$$

Then there exists a cone-valued function $V(t,x)$ with the following properties:

(I) $V \in C[\mathscr{R}^+ \times S(\rho), K], V(t,0) = 0$, *and $V(t,x)$ is locally Lipschitzian in x relative to K for each $t \in \mathscr{R}^+$, and for a continuous $\beta(t) > 0$,*
(II) $a[(\phi_0, x^*(t))] \leqslant (\phi_0, V(t,x)) \leqslant b[(\phi_0, x^*(t))]$ $a, b \in \mathscr{K}$ *and for* $\phi_0 \in K_0^*$ *and* $(t,x) \in \mathscr{R}^+ \times K$.
(III) $D^+(\phi_0, V(t,x)) \leqslant -c[(\phi_0, x^*(t))],\ c \in \mathscr{K}$.

Proof. From the hypotheses, solutions of the system (1.1) exist and are unique. Let $x(t, t_0, x_0)$ be a solution of (1.1) so that $x_0 = x(0, t, x)$. Define the function c as

$$c[(\phi_0, x^*(t))] = \frac{1}{A}[1 - \exp(1 - A(\phi_0, x^*(t)))],$$

where $A > 0$ is a constant. If $(\phi_0, x^*(t)) = 0$, then $\frac{1}{A}[1 - \exp(-A(\phi_0, x^*(t)))] = 0$. This implies that $c(0) = 0$. If $(\phi_0, x^*(t)) > 0$, then $\frac{1}{A}[1 - \exp(-A(\phi_0, x^*(t)))]$ is monotone increasing. It follows that $c \in K$. Now, we define a cone-valued Liapunov function $V(t,x)$ by

$$V(t,x) = \sup c[(\phi_0, x^*(t))]x(t + \delta, 0, \sigma_w(x(0,t,x)))\frac{1 + B\delta}{1 + \delta}, \tag{2.1}$$

where $\sigma_w : S(\rho) \to K$ is defined in [1] and $x^*(t)$ is the maximal solution of (1.1) relative to the cone $K \subset \mathscr{R}^n$. For $x = 0$, thus from (2.1), $V(t,0) = 0$, and for $\delta = 0$, we have

$$c[(\phi_0, x^*(t))]x(t + \delta, 0, \sigma_w(x(0,t,x))) \leqslant_K V(t,x).$$

Thus

$$c[(\phi_0, x^*(t))](\phi_0, x(t + \delta, 0, \sigma_w(x(0,t,x)))) \leqslant (\phi_0, V(t,x))$$

and

$$c[(\phi_0, x^*(t))]X_0(\phi_0, e) = a[(\phi_0, x^*(t))] \leqslant (\phi_0, V(t,x)), \tag{2.2}$$

where $X_0 = \min \mid x_i(t) \mid,\ i = 1, 2, \ldots, n,\ a(r) = u_0(\phi_0, e)c(r)$ and $e = (1, 1, \ldots, 1)^{\mathrm{T}}$. Since the zero solution of (1.1) is uniformly asymptotically ϕ_0-stable, then given $\epsilon > 0$, there exist two numbers $\delta = \delta(\epsilon)$, and $T = T(\epsilon)$ which are independent of t_0 such that

$$(\phi_0, x_0) < \delta \to (\phi_0, x^*(t)) < \epsilon, \quad \text{for } t \geqslant T(\epsilon).$$

By using the fact that $(1 + B\delta)/(1 + \delta) < B$ we get from (4.1) that

$$\begin{aligned}(\phi_0, V(t,x)) &= \sup_{\delta \geqslant 0} c[(\phi_0, x^*(t))][(\phi_0, x(t+\delta, 0, \sigma_w(x(0,t,x))))]\frac{1 + B\delta}{1 + \delta} \\ &\leqslant \sup_{\delta \geqslant 0} c[(\phi_0, x^*(t))][(\phi_0, x^*(t))]\frac{1 + B\delta}{1 + \delta} \\ &\leqslant B\epsilon c[(\phi_0, x^*(t))] \\ &= b[(\phi_0, x^*(t))],\end{aligned}$$

that is,

$$(\phi_0, V(t,x)) \leqslant b[(\phi_0, x^*(t))], \quad b \in \mathscr{K}. \tag{2.3}$$

Combining this with (2.2), we have

$$a[(\phi_0, x^*(t))] \leqslant (\phi_0, V(t,x)) \leqslant b(\phi_0, x(t)), \quad a, b \in \mathscr{K}. \tag{2.4}$$

This proves (II).

Now, for $\delta \geqslant T(\epsilon)$, where $T(\epsilon)$ is a monotonic decreasing function, we have from uniform asymptotic ϕ_0-stability that

$$(\phi_0, x^*(t)) < \epsilon.$$

Hence, if $\delta \geqslant T(\gamma(\phi_0, x^*(t)))$ for $\gamma > 0$, then

$$(\phi_0, x^*(t)) < \gamma(\phi_0, x^*(t))$$

implies

$$c[(\phi_0, x^*(t))] < c(\gamma(\phi_0, x^*(t))]$$

and

$$\begin{aligned}&c[(\phi_0, x^*(t))](\phi_0, x(t+\delta, 0, \sigma_w(x(0,t,u))))\frac{1 + B\delta}{1 + \delta} \\ &\quad \leqslant Bc[(\phi_0, x^*(t))](\phi_0, x(t)) \\ &\quad \leqslant b\epsilon c[\gamma(\phi_0, x^*(t))] \\ &\quad \leqslant (\phi_0, V(t,x)).\end{aligned}$$

Then

$$c[(\phi_0, x^*(t))]u(t+\delta, 0, \sigma_w(x(0,t,x)))\frac{1+B\delta}{1+\delta} \leqslant V(t,x).$$

This implies that $V(t,x)$ is defined only for $0 \leqslant \delta \leqslant T(\gamma(\phi_0, x^*(t)))$. As

$$V(t,x) = \sup_{0\leqslant\delta\leqslant T} c[(\phi_0, x^*(t))]u(t+\delta, 0, \sigma_w(x(0,t,x)))\frac{1+B\delta}{1+\delta},$$

$T = T(\gamma(\phi_0, x^*(t)))$.

By Corollary 2.7.1 of [2] and for $x_1, x_2 \in S(\rho)$, we have

$$\begin{aligned}
&\|V(t,x_1) - V(t,x_2)\| \\
&\quad = \left\| \sup_{0\leqslant\delta\leqslant T} c[(\phi_0, x^*(t))]x(t+\delta, 0, \sigma_w(x_1(0,t,x_1)))\frac{1+B\delta}{1+\delta} \right. \\
&\qquad \left. - \sup_{0\leqslant\delta\leqslant T} c[(\phi_0, x^*(t))]x(t+\delta, 0, \sigma_w(x_2(0,t,x_2)))\frac{1+B\delta}{1+\delta} \right\| \\
&\quad \leqslant \left| \sup_{0\leqslant\delta\leqslant T} c[(\phi_0, x^*(t))]\frac{1+B\delta}{1+\delta} \right| \|\sigma_w(x_1(0,t,x_1)) - \sigma_w(x_2(0,t,x_2))\| \\
&\quad \leqslant K(t,w)\left| \sup_{0\leqslant\delta\leqslant T} c[(\phi_0, x^*(t))]\frac{1+B\delta}{1+\delta} \right| \exp\int_0^t L(s)\mathrm{d}s \|x_1 - x_2\| \\
&\quad \leqslant \beta(t)\|x_1 - x_2\|,
\end{aligned}$$

where

$$\beta(t) = k(t,w)\left| \sup_{0\leqslant\delta\leqslant T} c[(\phi_0, x^*(t))]\frac{1+B\delta}{1+\delta} \right| \exp\int_0^t L(s)\mathrm{d}s$$

locally Lipschitzian in x_1 and x_2. Therefore $V(t,x)$ is locally Lipschitzian in x_1, x_2. Now

$$\begin{aligned}
\|V(t+\delta,x) - V(t,y)\| \leqslant{}& \|V(t+\delta,x) - V(t+\delta,y)\| \\
&+ \|V(t+\delta,y) - V(t+\delta, y(t+\delta,t,y))\| \\
&+ \|V(t+\delta,y) - V(t,y)\|. \qquad (2.5)
\end{aligned}$$

Since $V(t,y)$ is locally *Lipschitzian* in y and y is continuous in δ, then the first two terms in the right-hand side of the inequality (2.5) are small whenever $\|y-x\|$ and δ are small.

By using (2.1), the third term tends to zero as δ tends to zero. Therefore $V(t,x)$ is continuous in all its arguments.

Let $x = x(t,t_0,x_0), x_\rho = x(t+\rho,t,x), \rho > 0$. Then we have

$$V(t+\rho, x_\rho) = \sup_{0\leqslant\delta\leqslant T} c[(\phi_0, x^*(t))]x(t+\rho+\delta, 0, \sigma_w(x(0,t+\rho,u)))\frac{1+B\delta}{1+\delta}.$$

The continuity of V and the uniqueness of the solution of (1.1) imply that there exists a point δ_ρ in which the upper bound is reached so that we have

$$V(t+\rho,x_\rho) = c[(\phi_0,x^*(t))]x(t+\rho+\delta_\rho,0,\sigma_w(x(0,t+\rho,u)))\frac{1+B\delta}{1+\delta}.$$

By putting $\delta_\rho = \delta_1 - \rho$ and using the fact

$$\frac{1+B\delta_\rho}{1+\delta_\rho} = \frac{1+B\delta_1}{1+\delta_1}\left[1-\frac{(B-1)\rho}{(1+B\delta_1)(1+\delta_\rho)}\right]$$

we get

$$\begin{aligned} V(t+\rho,x_\rho) &= c[(\phi_0,x^*(t))]x(t+\rho,0,\sigma_w(x(0,t+\rho,u))) \\ &\quad\times\frac{1+B\delta_1}{1+\delta_1}\left[1-\frac{(B-1)_\rho}{(1+B\delta_1)(1+\delta_\rho)}\right] \\ &\leqslant_K V(t,x) - \frac{(B-1)_\rho V(t,x)}{(1+B\delta_1)(1+\delta_\rho)}. \end{aligned}$$

Since $0\leqslant\delta_\rho < T$, $0<\rho<\delta_1\leqslant\rho+T$, T is monotonically decreasing and using (2.4), we have

$$\frac{V(t+\rho,x_\rho)-V(t,x)}{\rho} \leqslant_K -\frac{(B-1)V(t,x)}{(1+B\delta_1)(1+\delta_\rho)},$$

$$\left(\phi_0,\frac{V(t+\rho,x_\rho)-V(t,x)}{\rho}\right)\leqslant_K -\frac{(B-1)(\phi_0,V(t,x))}{(1+B\delta_1)(1+\delta_\rho)}.$$

So

$$\begin{aligned} D^+(\phi_0,V(t,x)) &\leqslant -\frac{(B-1)(\phi_0,V(t,x))}{(1+BT(\gamma(\phi_0,x^*(t)))(1+T(\gamma(\phi_0,x^*(t)))+B\rho} \\ &\leqslant -\beta(\phi_0,V(t,x)),\quad \beta\in\mathscr{K}. \\ &\leqslant -\beta a[(\phi_0,x^*(t))]\leqslant -c(\phi_0,x^*(t))],\quad c\in\mathscr{K}. \end{aligned}$$

This proves (III), and the proof is completed. □

Theorem 2.2. *Let the hypotheses of Theorem* 2.1 *be satisfied. Then the zero solution of* (1.1) *is totally* ϕ_0*-stable.*

Proof. From Theorem 2.1, property (I) holds. Let $\epsilon>0$ be given, choose $\delta_1=\delta_1(\epsilon)$ such that

$$a(\epsilon) > b(\delta_1(\epsilon))\quad\text{for } a,b\in\mathscr{K}.$$

Let $x(t)=x(t,t_0,x_0)$ be any solution of (1.2) such that

$$(\phi_0,x_0)<\delta_1\quad\text{and}\quad(\phi_0,h(t,x))<\delta_2\ \text{ for } \delta_2=\delta_2(\epsilon)>0.$$

By condition (II) of Theorem 2.1, we have $V(t_0, x_0) = b(\delta_1(\epsilon))$. Now, we claim that

$$(\phi_0, V(t,x)) < a(\epsilon), \quad t \geqslant 0.$$

This claim leads to

$$a[(\phi_0, x^*(t))] \leqslant (\phi_0, V(t,x)) < a(\epsilon).$$

Then

$$(\phi_0, x^*(t)) < \epsilon.$$

This shows that the trivial solution of (1.1) is totally ϕ_0-stable. Now, we justify this claim. Define

$$T(t) = (\phi_0, V(t,x)),$$

and let this claim be false. Then there exist two numbers t_1 and t_2 with $t_0 < t_1 < t_2$ such that

$$T(t_1) = b(\delta_1(\epsilon)), \quad T(t_2) = a(\epsilon)$$

and

$$T(t) \geqslant b(\delta_1(\epsilon)) \quad \text{for } t_1 \leqslant t \leqslant t_2.$$

This shows that $T(t)$ is nondecreasing in $[t_1, t_2]$ and so we have

$$D^+T(t_1) \geqslant 0 \tag{2.6}$$

From (II) and (III) of Theorem 2.1 and for any $c^* \in \mathscr{K}$, we have

$$D^+(\phi_0, V(t,x)) \leqslant -c^*[(\phi_0, V(t,x))].$$

This implies that

$$\begin{aligned} D^+T &\leqslant -c^*(T) + M|(\phi_0, h(t,x))|, \quad M > 0 \\ &\leqslant -c^*(T) + M\delta_2 \\ &\leqslant -c^*(b(\delta_1(\epsilon))) + M\delta_2 \\ &= -b^*(\delta_1(\epsilon)) + M\delta_2, \end{aligned}$$

where $c^*(b(r)) = b^*(r) \in \mathscr{K}$. Now, choose $\delta_2 = b^*(\delta_1(\epsilon)/M)$. Then

$$D^+T < 0,$$

which contradicts (2.6) and our claim is justified. Therefore the zero solution is totally ϕ_0-stable and the proof is completed. □

Theorem 2.3. *Let the hypotheses of Theorem* 2.1 *be satisfied. Then the zero solution of* (1.1) *is totally* ϕ_0*-stable under permanent perturbation bounded in the mean.*

Proof. From Theorem 2.1, property (I) holds. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1.2) such that

$$(\phi_0, x_0) < \delta - 1 \text{ and } (\phi_0, h(t,x)) \leqslant \gamma(t), \text{ where } \int_{t_0}^{t_0+T^{\star}} \gamma(s)\mathrm{d}s < \delta_2.$$

Now, by proceeding as in the proof of Theorem 2.2, we arrive at the inequality (2.6). From (I) and (III) of the property (I) we have

$$\begin{aligned} D^+T &\leqslant -c^*(T) \leqslant -c^*(T) + M \mid (\phi_0, h(t,x)) \mid, \quad M > 0 \\ &\leqslant -c^*(T) + M\gamma(t). \end{aligned}$$

Integrating from t_0 to T^*, we get

$$\begin{aligned} T &\leqslant -\int_t^{T^*} c^*(T(s))\mathrm{d}s + M\int_{t_0}^{T^*} \gamma(s)\mathrm{d}s \\ &\leqslant -\int_{t_0}^{T^*} c^*(T(s))\mathrm{d}s + M\delta_2. \end{aligned}$$

Now, if we choose $\delta_2 = M^{-1}\int_{t_0}^{T^*} c^*(T(s))\mathrm{d}s$, then $T < 0$, that is

$$(\phi_0, V(t,x)) < 0.$$

But this is impossible since by the condition (II),

$$(\phi_0, V(t,x)) \geqslant a[(\phi_0, x^*(t))], \quad a \in \mathscr{K}.$$

Therefore the result is immediate. □

Theorem 2.4. *Let the conditions of Theorem* 2.1 *be satisfied, and further assume that* $h(t,x)$ *is locally Lipschitzain in* x *relative to the cone* $K \subset \mathscr{R}^n$ *for each* $t \in \mathscr{R}^+$. *Then the zero solution of* (1.1) *is uniformly totally* ϕ_0*-stable.*

Proof. From Theorem 2.1, it follows that

$$\begin{aligned} D^+(\phi_+0, V(t,x)) &\leqslant -c^*[(\phi_0, V(t,x))] + M[(\phi_0, h(t,x))] \\ &\leqslant -c^*[(\phi_0, V(t,x))] + M\sigma(t), \quad M > 0. \end{aligned}$$

Since $\sigma \in \mathscr{L}$, then there exists $T = T(\epsilon)$ sufficiently large such that for $t \geqslant T(\epsilon)$, we have that $\sigma(t) \to 0$. Therefore

$$D^+(\phi_0, V(t,x)) \leqslant -c^*(\phi_0, V(t,x)), \quad t \geq T(\epsilon).$$

From (II), we have

$$D^+(\phi_0, V(t,x)) \leqslant -c^*[(\phi_0, V(t,x)) + M\sigma(t) = -c^*[(\phi_0, x^*(t))].$$

where $c^*, a \in \mathscr{K}$ and $c^*[a(r)] = c(r)$ so that $c \in \mathscr{K}$. Now, using conditions (I), (II) and (III), we see that the conditions of Theorem 3.1 of [1] are satisfied. Since

$$\|h(t,x) - h(t,y)\| \leqslant L(t)\|x - y\| \quad \text{for } a, y \in K,$$

then putting $y = 0$, we get

$$\|h(t,x)\| \leqslant L(t)\|x\|,$$

when $x = 0$, we have $\|h(t,0)\| = 0$. Therefore from Theorem 3.4 of [1], and Definition 1.8 the result is immediated. □

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